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The Ising Model in a Random Magnetic Field

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The existence of a spontaneous magnetization in the three-dimensional Ising model in a weak random magnetic field (RFIM) is investgated. Following Imry and Ma, we consider the energy change, ΔE_γ, from the fully aligned ferromagnetic state caused by flipping all the spins inside a connected surface, γ. It is proved rigorously that with high probability, ΔE_γ is positive for *all* γ enclosing the origin. Under the unproven assumption that the expectation value of the spin at one site is weakly correlated with the random fields at far away sites (which is true if surfaces within surfaces can be ignored) it follows that the three-dimensional RFIM has a spontaneous magnetization at low temperatures. The proof works for all dimensions greater than two, providing support for the conjecture that two is the lower critical dimension.

KEY WORDS: Domain wall; ferromagnetic order, random field.

In recent years, the physics of disordered systems has been among the most active branches of statistical and condensed matter physics. Among the problems in this domain that have received much attention are the following:

1. Anderson localization and the metal–insulator transition.
2. The physics of glasses and of spin glasses.
3. Ferromagnets in a random magnetic field.

In this paper we describe and derive some mathematically precise results on the random field Ising model (RFIM), i.e., problem (3). The Ising model in a random magnetic field is used to describe many physical systems, among

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them the following:

a. Dilute, highly anisotropic antiferromagnets in a uniform, external magnetic field.[1,2]

b. The statistics of droplets of a fluid (e.g., oil) in a porous medium.[3]

c. Commensurate charge density waves in solids containing randomly distributed impurities.[4]

It is widely believed that the upper critical dimension of the RFIM is $d_u = 6$; see Refs. 5–7. There has, however, been considerable theoretical and experimental controversy as to the value of the lower critical dimension d_l, i.e., the dimension below which long-range ferromagnetic order cannot exist.[5–14] Perturbative arguments, involving a heuristic study of the most infrared divergent diagrams in a Landau–Ginsburg version of the RFIM, show that, to all orders in $6-\epsilon$, the critical behavior of the RFIM is the same as that of the pure Ising model in $4-\epsilon$.[5–7,10] This suggests that $d_l = 3$ since the lower critical dimension of the pure Ising model is one. These arguments can also be derived with the help of supersymmetry as noted by Parisi and Sourlas[10] and analyzed more carefully in Refs. 11 and 12. However, the first step in the Parisi–Sourlas argument in which thermal averages of $e^{-\beta H}$ are replaced by averages over *all* extrema of the Hamiltonian is questionable, since even at zero temperature where the approximation should be best, only absolute minima of the Hamiltonian should be included. In contrast, domain wall arguments of Imry and Ma,[13,14] which have much intuitive appeal, suggest that $d_l = 2$. In two dimensions one expects that there is no long-range ferromagnetic order at any temperature for arbitrarily weak disorder.

In this paper we propose a sharper and mathematically more precise version of the Imry–Ma domain wall arguments which lends additional support to the conjecture that $d_l = 2$.

Before proceeding to analyze the RFIM let us define the classical Hamiltonian function H which can be used to describe the statistics of droplets of a fluid in a porous medium:

$$H_\Lambda(n,h) = \alpha \sum_{i,j\in\Lambda}{}' (n_i - n_j)^2 + \epsilon \sum_{i\in\Lambda} h_i n_i \tag{1}$$

Here Λ is a subset of the lattice $\mathbb{Z}^d$. The sum $\sum'$ ranges over nearest-neighbor pairs in Λ, and the "occupation numbers" n_i take the values 0, or 1, for all i. The variables ϵh_i are energies which determine the affinity of the environment at site i to the fluid and α represents the interfacial tension. We suppose that the h_i are independent Gaussian random variables.

The remainder of this paper is devoted to analyzing the Hamiltonian for the RFIM equivalent to that in (1), given by

$$H_\Lambda(\sigma,h) = -\sum_{\Lambda}{}' \sigma_i\sigma_j - \epsilon \sum_{\Lambda} h_i\sigma_i \tag{2}$$

where $\sigma_i = \pm 1$. We shall assume again that the h_i are independent Gaussian and

$$\overline{h_i} = 0, \qquad \overline{h_i h_j} = \delta_{ij}$$

The bar denotes averages over the h. The parameter ϵ measures the strength of the disorder. Unless otherwise stated we shall always be considering the *three*-dimensional case. Let us denote the partition function and correlations with + (−) boundary conditions on the inner boundary of Λ by

$$Z_\Lambda^{+(-)}(h) = \sum_{\{\sigma\}}^{+(-)} e^{-\beta H_\Lambda(\sigma,h)} \tag{3}$$

$$\langle\sigma\rangle^{+(-)}(\beta,h) = \sum_{\{\sigma\}}^{+(-)} \sigma_i \frac{e^{-\beta H_\Lambda(\sigma,h)}}{Z_\Lambda^{+(-)}} \tag{4}$$

For $d > 2$ we shall present domain wall arguments in favor of a spontaneous magnetization, i.e.,

$$\overline{\langle\sigma_0\rangle^{+}(\beta,h)} > 1/2 \tag{5}$$

for all Λ containing any particular site 0, provided β is large and ϵ is small. Our approach is in the spirit of the coarse graining ideas of Grinstein and Ma[14] and also of Villain.[4] We consider *all possible connected domain walls*, even those which become disconnected after coarse graining. J. Chalker has presented arguments similar to those given here.[15] Under the assumption specified below we shall prove that for sufficiently large β and small ϵ,

$$\langle 1 - \sigma_0\rangle_\Lambda^{+}(\beta,h) \leqslant e^{-\beta} \tag{6}$$

holds with probability $1 - \exp(-\text{const}/\epsilon^2)$.[5] From this fact (5) follows easily.

To each spin configuration let us associate the set $\{\gamma_i\}$ where γ_i are *connected* domain walls or contours across which the spins σ change sign. Let $\bar\gamma$ be the set of sites inside γ. We let $|\gamma|$ and $|\bar\gamma|$ denote the surface area of γ and the volume of $\bar\gamma$, respectively. As in the Griffiths–Peierls argument[16] it is easy to show that

$$\langle 1 - \sigma_0\rangle_\Lambda^{+}(\beta,h) \leqslant \sum_{\bar\gamma \ni 0} e^{\beta F_{\bar\gamma}(h)} e^{-2\beta|\gamma|} \tag{7}$$

where the sum ranges over all contours (connected) which enclose the origin and

$$\beta F_{\bar\gamma}(h) \equiv \ln Z_{\bar\gamma}^{-}(h) - \ln Z_{\bar\gamma}^{+}(h) \tag{8}$$

The effect of the contours inside $\bar\gamma$ has been incorporated in our definition

[5] Throughout this paper "const" denotes some positive independent of γ, ϵ, β, and a scale index k introduced later.

of $F_{\bar{\gamma}}$. Clearly if the event

$$Q = \{ h \,|\, |F_{\bar{\gamma}}(h)| \leqslant |\gamma|, \text{ for all } \gamma, \bar{\gamma} \ni 0 \} \tag{9}$$

occurs with probability $1 - \exp(-\text{const}/\epsilon^2)$ then (6) follows from (7), for sufficiently large β, since the number of contours with area $|\gamma|$ is bounded above by $\exp(\text{const}|\gamma|)$, and $|\gamma| \geqslant 6$. See Ref. 16.

Roughly speaking $F_\gamma(h) = 2\epsilon \sum_{\bar{\gamma}} h_i$. More precisely, from (8) we have

$$F_{\bar{\gamma}}(h) = \int_0^1 \frac{d}{ds} F_{\bar{\gamma}}(sh)\, ds = \epsilon \sum_{i \in \bar{\gamma}} h_i \int_0^1 \langle \sigma_i \rangle (\beta, sh)\, ds$$

where

$$\langle \cdot \rangle (\beta, h) = \langle \cdot \rangle^- (\beta, h) + \langle \cdot \rangle^- (\beta, -h)$$

We shall establish the existence of a ferromagnetic state under the assumption that for $V_1, V_2 \subset \mathbb{Z}^3$

$$\text{Prob}\{ |F_{V_1}(h) - F_{V_2}(h)| \geqslant B \} \leqslant \exp\left(\frac{-\text{const}\, B^2}{\epsilon^2 |V_2 \setminus V_1|} \right) + \exp\left(\frac{-\text{const}\, B^2}{\epsilon^2 |V_1 \setminus V_2|} \right) \tag{10}$$

Here F_V is given by (8) with $\bar{\gamma}$ replaced by V.

If there are no contours within contours, as is assumed in Refs. 13, 14, and 15, then

$$F_{\bar{\gamma}}(h) = 2\epsilon \sum_{i \in \bar{\gamma}} h_i$$

and (10) follows from a simple Gaussian calculation. Also if $\langle \sigma_i \rangle (\beta, h)$ depends "weakly" on h_j for $|j - i| \gg 1$ then (10) holds. This weak dependence holds at each order of perturbation theory in the parameter ϵ since the connected correlations of the $(-)$ state decay exponentially fast. However, ϵ is a dangerous parameter to expand in, and a deeper analysis is needed to test the validity of (10).

To show that the complement of Q, Q^c, is unlikely we note that for a fixed contour γ

$$\text{Prob}\{ F_{\bar{\gamma}}(h) \geqslant |\gamma| \} \leqslant \exp \frac{-\text{const}|\gamma|^2}{\epsilon^2 |\bar{\gamma}|} \tag{11}$$

If $|\gamma| = L^2$ then $|\bar{\gamma}| \leqslant \text{const}\, L^3$, hence the right-hand side of (11) is bounded by $\exp(-\text{const}\, L/\epsilon^2)$. On the other hand there are[16] $\exp(\text{const}\, L^2)$ contours γ of area L^2, and we see that this is naive analysis is inadequate. We need to take advantage of the fact that many contours enclose essentially the same volume, i.e., the $\{F_{\bar{\gamma}}\}$ are highly dependent. This will be done by introducing coarse-grained contours made of lattice squares of side M^k,

$k = 0, 1, 2, \ldots$ which approximate γ. (The parameter M may be set equal to 2 for the purposes of our discussion.) The corresponding coarse-grained contours given by $\gamma = \gamma_0, \gamma_1, \gamma_2, \ldots$ are defined as follows.

Let $\mathscr{C}_k$ be the family of cubes with sides of length M^k, centered on the sublattice $M^k \cdot \mathbb{Z}^3$. A cube $c \in \mathscr{C}_k$ is said to be admissible with respect to γ if

$$|c \cap \bar{\gamma}| \geqslant \tfrac{1}{2} M^{3k}$$

Now we *define* $\bar{\gamma}_k$ to be the union of all admissible cubes $c \in \mathscr{C}_k$ and set $\gamma_k = \partial \bar{\gamma}_k$. For $k \geqslant 1$, $\bar{\gamma}_k$ need not be the interior of γ_k, in fact γ_k need not be connected.

Proposition 1. There are constants independent of k and γ such that

(a) $$|\gamma_k| < \text{const}|\gamma|$$

(b) $$|\bar{\gamma}_{k+1} \backslash \bar{\gamma}_k|, |\bar{\gamma}_k \backslash \bar{\gamma}_{k+1}| \leqslant \text{const}\, M^{k+1}|\gamma|$$

i.e., the volume of the corridor between $\bar{\gamma}_k$ and $\bar{\gamma}_{k+1}$ is of order $M^k|\gamma|$.

Sketch of Proof. Let $U = c \cup c'$, where c and c' are adjacent cubes in $\mathscr{C}_k$ with c admissible and c' inadmissible. To prove (a) it suffices to show that $|\gamma \cap U| \geqslant \text{const}\, M^{2k}$. By our definition of admissiblity we have $\frac{1}{2} M^{3k} \leqslant |\gamma \cap U| \leqslant \frac{3}{2} M^{3k}$. The lower bound assures us that $|\partial(\bar{\gamma} \cap U)| \geqslant \text{const}\, M^{2k}$. The upper bound may be used to show that $|\gamma \cap U| \geqslant \text{const}|\partial(\bar{\gamma} \cap U)|$. This can be seen by considering projections of γ onto the faces of U. To establish (b), consider a cube $\bar{c} \in \mathscr{C}_{k+1}$ such that $\bar{c} \cap \bar{\gamma}_{k+1} \backslash \bar{\gamma}_k \neq \emptyset$. This easily implies that $\bar{c}$ contains a pair of a cubes c, c' as above. Thus $|\bar{c}| \leqslant \text{const}\, M^k |\bar{c} \cap \gamma|$ and the first inequality of (b) follows. The second inequality is proved similarly.

Let us define the set of fields $\{h\}$ for which the total field in the corridor between $\bar{\gamma}_k$ and $\bar{\gamma}_{k+1}$ is not atypical:

$$\mathscr{E}_k(A) = \Big\{ h : |F_{\bar{\gamma}_k}(h) - F_{\bar{\gamma}_{k-1}}(h)| \leqslant A/(2k^2), \text{ for } all\ \gamma, \text{ with } |\gamma| = A \text{ and } \bar{\gamma} \ni 0 \Big\} \tag{13}$$

For fixed contour area A we terminate our coarse-graining procedure when $k = N(A) = N$, where $M^{2N(A)} \cong A^{2/3}$. The set of fields $\{h\}$ for which the final coarse-grained volume does not have an anomalously large total field

$$\mathscr{E}_N(A) = \Big\{ h : |F_{\gamma_N}(h)| < A/(2N^2) \text{ for all } \gamma, \text{ with } |\gamma| = A \text{ and } \bar{\gamma} \ni 0 \Big\} \tag{14}$$

Using the fact that $\sum_{k=1}^{\infty}(2k^2)^{-1} \leqslant 1$ it follows that

$$Q \supseteq \bigcap_{A \geqslant 6} \bigcap_{k \geqslant 1}^{N(A)} \mathscr{E}_k(A)$$

hence

$$\text{Prob}(Q^c) \leqslant \sum_{A \geqslant 6} \sum_{k}^{N(A)} \text{Prob}\{\mathscr{E}_k(A)^c\} \tag{15}$$

The next proposition is our main technical estimate.

Proposition 2. The number of coarse-grained contours γ_k as γ ranges over all possible connected contours γ such that $\bar{\gamma} \ni 0$ and $|\gamma| = A$ is less than

$$\exp\left(+ \frac{\text{const}\, kA}{M^{2k}} \right) \tag{16}$$

The idea behind the proof of this proposition is simple. By Proposition 1, γ_k consists of $\text{const}|\gamma|/M^{2k}$ squares of area M^{2k}. If γ_k were connected for all choices of γ, (16) would follow immediately. Although the coarse-grained contours need not be connected (even though γ is always connected), the number of components of γ_k is less than $\text{const}\, A/M^{2(k-1)}$ because each component must have $\text{const}\, M^{2(k-1)}$ points in it. This fact together with the observation that the distance connecting the components must be less than A (because γ is connected) enables us to establish Proposition 2. Details appear in the Appendix.

Next we claim that, for $k < N(A)$,

$$\text{Prob}\{\mathscr{E}_k(A)^c\} \leqslant \exp\left(\frac{2\,\text{const}\, Ak}{M^{2(k-1)}} \right) \cdot \exp\left(\frac{-\text{const}\, A}{\epsilon^2 k^4 M^k} \right)$$

By Proposition 2, the first factor is an upper bound on the number of coarse-grained pairs γ_k, γ_{k-1}. The second factor follows from our assumption (10) together with (13). It represents the probability that a particular coarse-grained corridor has a large total field. When $n = N(A)$ we note that $|\bar{\gamma}_N| \leqslant \text{const}\, A^{3/2}$. Thus

$$\text{Prob}\{\mathscr{E}_N(A)^c\} \leqslant \exp\left[2 - \text{const}\, A^{1/3} N(A)\right] \cdot \exp \frac{-\text{const}\, A^{1/2}}{\epsilon^2 N^4(A)}$$

which is small because $N(A)$ is logarithmic in A.

Now the sum over A and k in (15) is easily seen to converge and yield the bound

$$\text{Prob}\{ Q^c \} < \exp(-\text{const}/\epsilon^2)$$

This completes our proof. Formally similar arguments can be made in $2 + \epsilon$ dimensions for any positive ϵ. However, when $d = 2$ the bound for the number of pairs γ_k, γ_{k-1} becomes

$$\exp\left[2\,\mathrm{const}\,Ak/M^{k-1}\right]$$

causing the sum in (15) to diverge, and our proof fails.

We hope that our assumption (10) can be proved for three dimensions and that in fact the arguments given here will provide an initial step. However, in two dimensions it will quite possibly break down, and contours within contours may play an important role.

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APPENDIX: PROOF OF PROPOSITION 2

Let γ_k^α denote the components of γ_k, i.e., the separate connected parts of γ_k. As we noted earlier, there are less than $\mathrm{const}\,A/M^{2(k-1)} \equiv \alpha^*$ such components, and $\sum_\alpha |\gamma_k^\alpha| \leqslant \mathrm{const}\,A$. Let x^α be a set of lattice points in $M^k \cdot \mathbb{Z}^3$, and define $d_1 = |x^1|$, $d_2 = |x^2 - x^1|, \ldots, d_l = |x^l - x^{l-1}|$. Now fix x^α, $\alpha = 1, 2, \ldots \leqslant \alpha^*$. Let $\Gamma_k(\{x^\alpha, a^\alpha\})$ be the number of coarse-grained contours γ_k such that each γ_k^α contains a fixed point x^α and has area $a^\alpha = |\gamma_k^\alpha|$. Then

$$\Gamma_k(\{x^\alpha, a^\alpha\}) \leqslant \exp\left(\mathrm{const} \sum a^\alpha / M^{2k}\right) \leqslant \exp(\mathrm{const}\,A/M^{2k}) \quad \text{(A.1)}$$

The number of $\{a^\alpha\}$ which are multiples of M^{2k} such that $\sum a^\alpha \leqslant A$ is less, than $2^{A/M^{2k}}$.

It now remains to bound the number of possible x^α's. Since γ is connected, it follows that we can relabel the α such that

$$\sum_\alpha^{\alpha^*} d_\alpha \leqslant 2A \quad \text{(A.2)}$$

If the d_α are specified it is easy to see that there are less than $\prod_\alpha^{\alpha^*} \mathrm{const}\, d_\alpha^3$ possible choices of the $\{x^\alpha\}$ in three dimensions. This product is maximal when the d_α are equidistant; hence the product is bounded by

$$\left|\frac{\mathrm{const}\,A}{\alpha^*}\right|^{3\alpha^*} \leqslant \exp\left|\mathrm{const}\,k\,\frac{A}{M^{2k}}\right|$$

Finally the number of solutions to (A.2) is also bounded by $|(\mathrm{const}\,A)/\alpha^*|^{\alpha^*}$, and our proof of Proposition 2 is complete.

Remark. We have used the fact that the number of solutions $(d_1, \ldots, d_m)$ to the equation

$$\sum_{i=1}^{m} d_i = N, \qquad d_i \geqslant 1$$

equals the binomial coefficient

$$\binom{N-1}{m} \leqslant \frac{N^m}{m!} \leqslant \left(\frac{\text{const}\, N}{m}\right)^m$$

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